Problems from distance geometry

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Theorem 1 (Erdős–Anning theorem [1]). *An infinite number of points in the plane can have mutual integer distances only if all the points lie on a straight line.*

The article [1] also contains some examples for infinite point sets with pairwise rational distances. In all of these examples and in every other known example there are only constantly many point not lying on a line or on a circle.

Problem 2 (Erdős-Ulam). Is there a dense set in the plane such that all pairwise Euclidean distances are rational?

It would be very surprising if this would be true. It would imply for example that the ABC conjecture is false. As the following problem shows we are very far from finding an appropriate dense set.

Problem 3 (Erdős). Is there a point set with n points such that the pairwise distances are integers, no three point lies on a line and no four lies on a circle? (So far the record is 7 points, see [2])

Theorem 4 (Berry). *Given a triangle where one of sides have rational length and the other two lengths are either rational or square roots of a rational number. Then there are infinitely many points in the plane that are at rational distance from the three vertices of the triangle.*

Problem 5. Is there a point in the plane that is at rational distance from the vertices of the unit square?

By Theorem 4 we can find points that are at rational distance from three of the vertices of the square.

Let G^1 be the graph whose vertices are the points of $\mathbb{R}^2$ and two points are connected if their distance is 1. Similarly let G^{odd} be the graph whose vertices are the points of $\mathbb{R}^2$ and two points are connected if their distance is an odd integer. These graphs are known as the unit distance graph and the odd distance graph.

Problem 6. What is the chromatic number of G^1?

The best bounds we know is $5 \leq \chi(G^1) \leq 7$.

Problem 7. What is the chromatic number of G^{odd}?

Since $\chi(G^{odd}) \geq \chi(G^1)$ we know that $\chi(G^{odd}) \geq 5$ but we don't even now if $\chi(G^{odd})$ is finite. If the color classes are measurable, then finitely many colors are not enough ([10] and [6]).

Problem 8. Which graphs are subgraphs of G^1? Which are subgraphs of G^{odd}?

We know that $K_{n,n,n}$ is a subgraph of G^{odd}, hence every 3-colorable graph is a subgraph. This follows from a construction that can be found in [7].

It is not hard to show that K_4 is not a subgraph of G^{odd}, and it has also been shown that the 5-wheel graph is also not a subgraph ([8]).

Problem 9 (Harborth's conjecture)**.** Is it true that every planar graph has a planar drawing in which every edge is a straight segment of integer length?

Note that Problem 2 would immediately imply Harborth's conjecture. We could take any geometric drawing of the graph and we could perturb the vertices till all the distance are rational. Then we could enlarge the drawing to get integer distances.

Lemma 10. *Let P be a finite set of points in the plane. Assume that the distance between any two points is an integer. Prove that P can be colored by three colors such that the distance between any two points of the same color is an even number. (2018 Schweitzer Competition Problem 4, Gábor Damásdi)*

Corollary 11. *For any k there is n_k such that following holds. Let P be a finite set of n_k points in the plane where the distance between any two points is an integer. Then one of the distances is divisible by 2^k.*

Problem 12. Can we generalize Corollary 11 to being divisible by p^k for any p prime?

Problem 13. For which p primes is it true that we can draw K_n for any n with integer distances that are not divisible by p. For example K_4 cannot be drawn with odd distances, so 2 is not good.

Problem 14. Give new ways of constructing point sets with pairwise integer distances. (It is known that no irreducible algebraic curve other than a line or a circle contains an infinite rational set.)

Problem 15. A Robbins pentagon is a cyclic pentagon whose side lengths and area are all integer numbers. It is known that the diagonals of a Robbins pentagon are either all rational or all irrational ([9]). Is there a Robbins pentagon where all the diagonals are irrational?

References

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